Deformations of holomorphic pairs and Scattering diagrams

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Abstract

Scattering diagrams first appear in mirror symmetry [KS] as combinatorial data which prescribe how to reconstruct the mirror manifold. In particular it has been proved that on one hand they encode Gromov–Witten invariants [GPS] and, on the other hand that they govern deformations of the complex structure [CLM]. In [Fan] the author introduced the extended tropical vertex group $\tilde{\mathbb{V}}$ by studying the asymptotic behaviour of certain special solutions to the Maurer-Cartan equation which govern infinitesimal deformations of a semi-flat Calabi-Yau manifold together with a holomorphically trivial vector bundle on it. The main result is that the leading order asymptotics defines naturally a consistent scattering diagram in the new group $\tilde{\mathbb{V}}$.

Setting

Let B be a tropical affine manifold, Λ be a lattice subbundle of the tangent bundle TB and $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the dual lattice, which is a subbundle of the contanget bundle T^*B.

SYZ fibration

B-model

$\check{X} := TB/\Lambda$ is the total space of the torus fibration $\check{p}: \check{X} \to B$ endowed with the one parameter complex structure $J_\hbar$, $\hbar \in \mathbb{R}_{>0}$, induced from the complex structure of TB.

A-model

$X := T^*B/\Lambda^*$ is the total space of the dual torus fibration $p: X \to B$, which is a symplectic manifold with symplectic form $\omega_\hbar := \hbar^{-1}\omega$, where ω is the canonical complex structure on X.

Let $E \to \check{X}$ be a rank r holomorphically trivial vector bundle on $\check{X}$ and fix an hermitian metric h which is constant along the fibers of $\check{X}$. Moreover, we set F_E the Chern curvature of the Chern connection ∇^E of $(E, h, \bar\partial_E)$.

Deformations of $(E, \check{X})$

Let $\mathbf{A}(E) := \mathrm{End}\, E \oplus T^{1,0}\check{X}$ be the holomorphic bundle on $\check{X}$, endowed with the complex structure $\bar\partial_{\mathbf{A}(E)} := \begin{pmatrix} \bar\partial_E & \mathbf{B} \\ 0 & \bar\partial_{\check{X}} \end{pmatrix}$, where $\mathbf{B}\varphi := \varphi \lrcorner F_E$ for every $\varphi \in \Omega^p(\check{X}, T^{1,0}\check{X})$. The simbol $\lrcorner$ denotes the contraction of forms with vector fields. The Kodaira–Spencer DGLA which governs infinitesimal deformations of the pair $(E, \check{X})$ is

$$\mathsf{KS}(E, \check{X}) := \left(\Omega^{0,\bullet}(\check{X}, \mathbf{A}(E)), \bar\partial_{\mathbf{A}(E)}, [\cdot,\cdot]\right)$$

where the Lie bracket

$$[\cdot,\cdot]\colon \Omega^{0,p}(\mathbf{A}(E)) \times \Omega^{0,q}(\mathbf{A}(E)) \to \Omega^{0,p+q}(\mathbf{A}(E))$$

is defined as follows

$$[(A,\varphi),(A',\psi)] := \big([A,A'] + \varphi \lrcorner \nabla^E A' + \\ -(-1)^{pq}\psi \lrcorner \nabla^E A, [\varphi,\psi]\big)$$

Infinitesimal deformations of the pair $(E, \check{X})$ are $(A, \varphi) \in \Omega^{0,1}(\check{X}, \mathbf{A}(E))[\![t]\!]$ which are solutions of **Maurer–Cartan equation**

$$\bar\partial_{\mathbf{A}(E)}(A,\varphi) + \frac{1}{2}[(A,\varphi),(A,\varphi)] = 0 \quad \text{[MC]}$$

up to gauge equilvance, where the action of gauge group is defined by $h \in \Omega^0(\check{X}, \mathbf{A}(E))[\![t]\!]$

$$\exp(h) * (A,\varphi) := (A,\varphi) + \\ -\sum_{k\geq 0} \frac{\mathsf{ad}_h^k}{(k+1)!}\left(\bar\partial_{\mathbf{A}(E)}h + [(A,\varphi),h]\right)$$

Scattering Diagrams in the Extended Tropical Vertex

Let $\mathbb{C}[\![t]\!]$ be the ring of formal power series in the parameter t. Elements of $n \in \Lambda^*$ can be regarded as derivations $\partial_n \in \mathrm{Der}(\mathbb{C}[\Lambda])$ where $\partial_n z^m := \langle n, m\rangle z^m$. Let $\tilde{\mathfrak{h}}$ be the Lie algebra

$$\tilde{\mathfrak{h}} := \bigoplus_{m \in \Lambda \setminus \{0\}} \mathbb{C}z^m\left(t \cdot (\mathfrak{gl}(r,\mathbb{C}) \oplus m^\perp)\right) \subset \left((t)\mathfrak{gl}(r, \mathbb{C}[\Lambda])\hat\otimes_{\mathbb{C}}\mathbb{C}[\![t]\!]\right) \oplus \left((\mathbb{C}[\Lambda]\hat\otimes_{\mathbb{C}}\mathbb{C}[\![t]\!])(t) \otimes_{\mathbb{Z}} \Lambda^*\right)$$

$$[(Az^m, z^m\partial_n), (A'z^{m'}, z^{m'}\partial_{n'})] := \big([A,A']_{\mathfrak{gl}} z^{m+m'} + A'\langle n, m'\rangle z^{m+m'} - A\langle n', m\rangle z^{m+m'}, \\ z^{m+m'}\partial_{\langle n,m'\rangle m' - \langle n',m\rangle n}\big)$$

where $m^\perp \in \Lambda^*$ denotes ∂_n with the unique primitive vector $n \in \Lambda^*$ such that $\langle n, m\rangle = 0$ and it is positively oriented with respect to $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. The **extended tropical vertex group** $\tilde{\mathbb{V}} := \exp(\tilde{\mathfrak{h}})$ is a subgroup of $GL(r, \mathbb{C}[\Lambda][\![t]\!]) \times \mathrm{Aut}_{\mathbb{C}[\![t]\!]}(\mathbb{C}[\Lambda][\![t]\!])$ with the BCH product.
Scattering diagrams are defined as a collection of *walls* $\mathbf{w}_i = (\mathfrak{d}_i, \theta_i)$:

- $\mathfrak{d}_i$ is either a *ray* ($\mathfrak{d}_i = \xi_0 + \mathbb{R}_{\geq 0}m_i$) or a *line* ($\mathfrak{d}_i = \mathbb{R}m_i$), where $m_i \in \Lambda$ and $\xi_0 \in \Lambda_{\mathbb{R}}$;
- $\theta_i \in \tilde{\mathbb{V}}$, such that $\log\theta_i := \sum_{ij}\left(A_{jk}z^{km}t^j, a_{jk}z^{km}t^j\partial_n\right) \in \tilde{\mathfrak{h}}$.

  Moreover, for every $N > 0$ there are finitely many walls with $\theta_i \neq 1 \bmod t^N$.

The above defintion is modelled to definition of scattering diagrams of [GPS], where the tropical vertex is replaced by $\tilde{\mathbb{V}}$. Given a generic loop $\gamma\colon [0,1] \to \Lambda_{\mathbb{R}}$ the *path ordered product* $\Theta_{\gamma,\mathfrak{D}}$ is defined as the product of the θ_i according to the order of the path γ. In particular, $\mathfrak{D}$ is a **consistent** scattering diagram if for every generic loop γ, $\Theta_{\gamma,\mathfrak{D}} = \mathrm{id}_{\tilde{\mathbb{V}}}$.

Theorem ([KS]). *Let $\mathfrak{D}$ be a scattering diagram. There exists a unique minimal scattering diagram $\mathfrak{D}_\infty \setminus \mathfrak{D}$ such that $\mathfrak{D}_\infty \setminus \mathfrak{D}$ consists only of rays, and it is* consistent.

Main Result

Theorem ([Fan]). *Let $\mathfrak{D} = \{\mathsf{w}_1, \mathsf{w}_2\}$ be a scattering diagram with non parallel walls. Then the consistent scattering diagram $\mathfrak{D}_\infty$ can be computed from the the asymptotic behaviour of solutions of MC equation.*

Proof:

We briefly sketch the proof of the main result:
▶ $(\tilde{\mathfrak{h}}, [\cdot,\cdot]_{\tilde{\mathfrak{h}}})$ is a Lie sub-algebra of $(\Omega^0(\mathbf{A}(E)), [\cdot,\cdot]_{\mathsf{KS}})$, in the limit $\hbar \to 0$. Let $\mathscr{F}$ be the Fourier transform acting on the fibers of $\check{X}$ then $\mathfrak{w}^m := \mathscr{F}(e^{2\pi i(m,z)}) \in \Omega^0(U)$ and $\partial_n := \mathscr{F}(\check\partial_n) \in \Omega^0(TB)$ where $\check\partial_n \in \Omega^0(T^{1,0}\check{X})$. In addition z^m gets replaced by $\mathfrak{w}^m$ in the definition of $\tilde{\mathfrak{h}}$.
Furthermore, $[\cdot,\cdot]_{\tilde{\mathfrak{h}}}$ is the leading order term in $\hbar$ of $\mathscr{F}([\mathscr{F}^{-1}(\cdot), \mathscr{F}^{-1}(\cdot)]_{\mathsf{KS}})$.
▶ $\mathfrak{D} \rightsquigarrow \Pi := \Pi_1 + \Pi_2$, where Π_1, Π_2 are solutions of MC respectively supported on $\mathfrak{d}_1$, $\mathfrak{d}_2$, i.e. for $i = 1, 2$ there exists a unique gauge h_i so that $\Pi_i = \exp(h_i) * 0$ and $\lim_{\hbar\to 0} h_i = \log(\theta_i)$ on the upper half plane above $\mathfrak{d}_i$ and it vanishes otherwise.
▶ $\Phi := \Pi + \sum_{a \in \mathbb{Z}^2_{\mathrm{prim}}} \Phi^{(a)}$ is a solution of MC. Furthermore for every $a \in \mathbb{Z}^2_{\mathrm{prim}}$ $\Phi^{(a)}$ is a solution of MC supported on a ray of slope $m_a := a_1 m_1 + a_2 m_2$, namely there is a unique h_a such that $\exp(h_a) * 0 = \Phi^{(a)}$.
▶ $\Phi \rightsquigarrow \mathfrak{D}_\infty := \mathfrak{D} \cup \{\mathsf{w}_a = (\mathfrak{d}_a, \theta_a)\}$, where $\mathfrak{d}_a := m_a\mathbb{R}_{\geq 0}$ and $\log\theta_a := \lim_{\hbar\to 0} h_a$
▶ $\Theta_{\gamma,\mathfrak{D}_\infty} = \mathrm{id}_{\tilde{\mathbb{V}}}$ which is proved by a monodromy argument.

Conclusion

∗*On the B-side*, our main result shows the relation between scattering diagrams in $\tilde{\mathbb{V}}$ and the asymptotic behaviour of infinitesimal deformations of $(E, \check{X})$.
∗*On the A-side*, in [Fan2] the author proves that certain genus zero Gromov–Witten invariants for toric surfaces relative to their toric boundary divisor, can be computed from consistent scattering diagrams in $\tilde{\mathbb{V}}$. However, we do not have a complete description of the enumerative contributions coming from the mirror of $(E, \check{X})$.
∗Finally, in [Fan] the author proves that scattering diagrams in $\tilde{\mathbb{V}}$ have *applications in physics* with 2*d*-4*d* wall crossing formula.

References

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